

# Derived manifolds and $d$ -manifolds

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## Abstract

A model structure is defined on the category of derived  $C^\infty$ -schemes, and it is used to analyse the truncation 2-functor from derived manifolds to  $d$ -manifolds. It is proved that the induced 1-functor between the homotopy categories is full and essentially surjective, giving a bijection between the sets of equivalence classes of objects. An example is constructed, showing that this 1-functor is not faithful.

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## 1 Introduction

Derived  $C^\infty$ -manifolds were defined in [Sp10] as topological spaces, equipped with homotopy sheaves of homotopy simplicial  $C^\infty$ -rings, s.t. locally they are weakly equivalent to 0-loci of smooth sections of vector bundles over manifolds.

In [BN11] it was shown that, using softness of the structure sheaves, one gets a homotopically equivalent construction without homotopy sheaves and with strict simplicial  $C^\infty$ -rings, instead of the homotopy ones.

In either one of these approaches to derived  $C^\infty$ -geometry the method is to use the homotopical algebra of simplicial  $C^\infty$ -rings to construct the “correct” 0-loci of sections of bundles, and then to glue them by weak equivalences.

There is another way of doing derived geometry. Instead of using algebraic homotopy theory, one can construct a 2-category of derived objects directly out of the geometry of manifolds, bundles and sections.

The recipe is as follows: start with a manifold  $\mathcal{M}$ , a vector bundle  $E$  over  $\mathcal{M}$ , and a section  $\sigma : \mathcal{M} \rightarrow E$ . Then take  $E^*$ , restrict it to the locus  $\{\sigma = 0\}$ , and consider the exact sequence

$$E^*|_{\{\sigma=0\}} \longrightarrow \mathcal{O}'_{\{\sigma=0\}} \longrightarrow \mathcal{O}_{\{\sigma=0\}} \longrightarrow 0, \quad (1)$$

where  $\mathcal{O}'_{\{\sigma=0\}}$  is the structure sheaf of the first infinitesimal neighbourhood of  $\{\sigma = 0\}$  in  $\mathcal{M}$ . This is an example of a  $d$ -manifold ([Jo12]).  $D$ -manifolds are then organised into a 2-category, which extends the category of usual manifolds and allows one to define virtual fundamental classes.

In this paper we compare these two theories of derived  $C^\infty$ -geometry. There is a straightforward truncation 2-functor from derived manifolds to  $d$ -manifolds, which is given by a linearization (in the vertical direction on  $E$ ) of the usual normalized complex of simplicial  $C^\infty$ -rings.

This truncation induces a 1-functor between the homotopy categories, and we show that this 1-functor is full, essentially surjective and gives a bijection between the equivalence classes of objects. However, it is not faithful, since by linearising we lose a lot of homotopically non-trivial information.

To analyse this truncation we develop a homotopy theory of derived manifolds, which can be of independent interest. More precisely, we construct a right pseudo-model category structure on the category of derived  $C^\infty$ -schemes of finite type. This model structure allows us to compute all the necessary homotopy pullbacks and mapping spaces.

Here is the structure of the paper. In section 2.1 we recall the definition of cosimplicial  $C^\infty$ -schemes and define the right pseudo-model structure on the subcategory of schemes of finite type.

In section 2.2 we show that this right pseudo-model category can be supplemented with a right action of the category of simplicial sets, thus giving us the ability to compute mapping spaces.

In section 2.3 we recall the definition of derived manifolds, and compute explicitly the simplicial  $C^\infty$ -rings of the 0-loci of sections of vector bundles over manifolds of finite type. The result is rather simple:  $\{C^\infty(E^{\times^k \mathcal{M}})\}_{k \geq 0}$ .

In section 3.1 we recall the definition of the 2-category of  $d$ -manifolds from [Jo12]. Our presentation is slightly different from the original: instead of pulling the sheaves back, we push them forward, and instead of using morphisms out of the sheaves of differential forms we use derivations.

In section 3.2 we describe the truncation 2-functor. Here the simplicial model structure, that we have developed for derived manifolds, comes very handy.

Finally, in section 3.3 we prove the main result: the truncation functor induces a full and essentially surjective 1-functor between the homotopy categories. The sets of equivalence classes of objects are in bijective correspondence. At the end of the section we provide an example, showing that this 1-functor is far from being faithful.

## 2 Derived manifolds

### 2.1 Cosimplicial $C^\infty$ -schemes

Recall (e.g. [MR91]) that a  $C^\infty$ -ring consists of a set  $A$ , together with operations  $A^{\times^n} \rightarrow A$  for all  $n \geq 0$ , parameterized by smooth functions  $\mathbb{R}^n \rightarrow \mathbb{R}$ . Equivalently, a  $C^\infty$ -ring is given by a product preserving functor  $\mathfrak{E} \rightarrow \text{Set}$ , where  $\mathfrak{E}$  is the category, having  $\{\mathbb{R}^n\}_{n \geq 0}$  as objects, and smooth maps as morphisms.

A morphism of  $C^\infty$ -rings  $A \rightarrow B$  is a set theoretic map, that is compatible with the action of smooth functions. A simplicial  $C^\infty$ -ring is just a simplicial diagram in the category of  $C^\infty$ -rings, and morphisms of simplicial  $C^\infty$ -rings are natural transformations. We will denote the categories of  $C^\infty$ -rings and simplicial  $C^\infty$ -rings by  $C^\infty\mathcal{R}$  and  $SC^\infty\mathcal{R}$  respectively.

**Definition 1** *Let  $A_\bullet$  be a simplicial  $C^\infty$ -ring. The spectrum of  $A_\bullet$ , denoted by  $\mathbf{Sp}(A_\bullet)$ , is the pair  $(\text{Sp}(A_\bullet), \mathcal{O}_{\bullet, \text{Sp}(A_\bullet)})$ , where*

$$\text{Sp}(A_\bullet) := \text{Hom}_{C^\infty\mathcal{R}}(\pi_0(A_\bullet), \mathbb{R}), \quad (2)$$

*together with Zariski topology, and  $\mathcal{O}_{\bullet, \text{Sp}(A_\bullet)}$  is the sheaf of simplicial  $C^\infty$ -rings on  $\text{Sp}(A_\bullet)$ , obtained by the sheafification of*

$$U \longmapsto \{A_n\{(s^n U)^{-1}\}\}_{n \geq 0}, \quad (3)$$

where  $\mathcal{U} := \{f \in A_0 \text{ s.t. } p([f]) \neq 0 \ \forall p \in U\}$ ,  $s^n : A_0 \rightarrow A_n$  is the  $n$ -fold degeneration, and  $A_n\{(s^n\mathcal{U})^{-1}\}$  denotes localization in the category of  $C^\infty$ -rings ([MR91]).

If  $A_\bullet$  is a discrete simplicial finitely generated  $C^\infty$ -ring, i.e. a constant simplicial diagram on  $A = C^\infty(\mathbb{R}^n)/\mathfrak{A}$ , then  $\mathbf{Sp}(A_\bullet)$  is just  $(X, \mathcal{O}_X)$ , where  $X \subseteq \mathbb{R}^n$  is the set of zeroes of  $\mathfrak{A}$ , and  $\mathcal{O}_X$  is the sheaf of germs of smooth functions around  $X \subseteq \mathbb{R}^n$ , modulo germs of functions in  $\mathfrak{A}$ .

There is another way of defining the spectrum of a simplicial  $C^\infty$ -ring  $A_\bullet$ . Instead of having one topological space  $\text{Hom}_{C^\infty\mathcal{R}}(\pi_0(A_\bullet), \mathbb{R})$  equipped with a sheaf of simplicial  $C^\infty$ -rings, one can have a cosimplicial diagram of  $C^\infty$ -schemes:

$$\{\text{Hom}_{C^\infty\mathcal{R}}(A_n, \mathbb{R}), \mathcal{O}_{Sp(A_n)}\}_{n \geq 0}. \quad (4)$$

However, we would like to consider such cosimplicial diagrams as weakly equivalent, if they have similar structure in the neighbourhood of  $Sp(\pi_0(A_\bullet))$  inside  $Sp(A_0)$ . Thus it is better to work with only one underlying topological space, and we have the following definition.

**Definition 2** A cosimplicial  $C^\infty$ -scheme is a pair  $(X, \mathcal{O}_{\bullet, X})$ , where  $X$  is a topological space, and  $\mathcal{O}_{\bullet, X}$  is a sheaf of simplicial  $C^\infty$ -rings, s.t. locally  $(X, \mathcal{O}_{\bullet, X})$  is isomorphic to spectra of simplicial  $C^\infty$ -rings.

Notice that for any  $p \in X$ , the stalk  $(\mathcal{O}_{0, X})_p$  is a local  $C^\infty$ -ring.<sup>1</sup> Alternatively, one can define a cosimplicial  $C^\infty$ -scheme as a simplicially  $C^\infty$ -ringed space, that is locally *weakly equivalent* to spectra of simplicial  $C^\infty$ -rings ([BN11], [Sp10]). In this case only  $(\pi_0(\mathcal{O}_{\bullet, X}))_p$  would be local. In the important situations ( $\pi_0(\mathcal{O}_{\bullet, X})$  being locally of finite type) these two definitions are locally weakly equivalent ([BN11]).

We will say that a simplicial  $C^\infty$ -ring  $A_\bullet$  is of finite type if  $\pi_0(A_\bullet)$  is a finitely generated  $C^\infty$ -ring. Such rings have particularly nice spectra, as the following proposition shows.

**Proposition 1** ([BN11]) *Let  $A_\bullet$  be a simplicial  $C^\infty$ -ring of finite type. Then  $Sp(A_\bullet)$  is homeomorphic to a locally closed subset of  $\mathbb{R}^n$  for some  $n \geq 0$ ,  $\pi_0(\mathcal{O}_{\bullet, Sp(A_\bullet)})$  is a sheaf of finitely generated  $C^\infty$ -rings, and  $\mathcal{O}_{0, Sp(A_\bullet)}$  is soft.*

We will work only with simplicial  $C^\infty$ -rings of finite type. We will also require that the underlying spaces of the cosimplicial  $C^\infty$ -schemes are second countable and Hausdorff. All these conditions together imply softness of the structure sheaves. This explains the following definition of separability.

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<sup>1</sup>A  $C^\infty$ -ring  $A$  is local, if it has a unique maximal ideal  $\mathfrak{A} \subset A$ , and  $A/\mathfrak{A} \cong \mathbb{R}$ .

**Definition 3** • A cosimplicial  $C^\infty$ -scheme  $(X, \mathcal{O}_{\bullet, X})$  is separated if  $X$  is Hausdorff, and  $\mathcal{O}_{0, X}$  is soft.

- A cosimplicial  $C^\infty$ -scheme  $(X, \mathcal{O}_{\bullet, X})$  is of finite type, if  $\pi_0(\mathcal{O}_{\bullet, X})$  is a sheaf of finitely generated  $C^\infty$ -rings.
- A cosimplicial  $C^\infty$ -scheme is locally of finite type, if locally it is isomorphic to a  $C^\infty$ -scheme of finite type.

We will denote the category of cosimplicial  $C^\infty$ -schemes by  $\mathbf{C}^\infty\mathbf{Sch}$ .<sup>2</sup> We will say that  $(X, \mathcal{O}_{\bullet, X}) \in \mathbf{C}^\infty\mathbf{Sch}$  is compact, second countable etc., if the space  $X$  is such. We will denote by  $\mathbf{G} \subset \mathbf{C}^\infty\mathbf{Sch}$  the full subcategory, consisting of separated, second countable cosimplicial  $C^\infty$ -schemes, locally of finite type. We will also denote by  $\mathbf{G}_{ft} \subset \mathbf{G}$  the full subcategory of cosimplicial  $C^\infty$ -schemes of finite type.

The following proposition shows that, just as in the usual  $C^\infty$ -geometry, most cosimplicial  $C^\infty$ -schemes are affine.

**Proposition 2** ([BN11]) Let  $\mathcal{G}_{ft} \subset SC^\infty\mathcal{R}$  be the full subcategory, consisting of simplicial  $C^\infty$ -rings of finite type. Then  $\Gamma \circ \mathbf{Sp} : \mathcal{G}_{ft}^{op} \rightarrow \mathcal{G}_{ft}^{op}$  maps weak equivalences to weak equivalences, and in the adjunction

$$\Gamma : \mathbf{G}_{ft} \rightleftarrows \mathcal{G}_{ft}^{op} : \mathbf{Sp} \quad (5)$$

the unit  $Id_{\mathbf{G}_{ft}} \rightarrow \mathbf{Sp} \circ \Gamma$  is an isomorphism.

Proposition 2 realises  $\mathbf{G}_{ft}$  as a full co-reflective subcategory of  $\mathcal{G}_{ft}^{op}$ . The latter is a full subcategory of the category  $SC^\infty\mathcal{R}^{op}$  of all simplicial  $C^\infty$ -rings, and it is well known ([Qu67]) that  $SC^\infty\mathcal{R}$  is a model category. A morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$  is a weak equivalence if it induces bijections

$$\pi_n(A_\bullet) \xrightarrow{\cong} \pi_n(B_\bullet), \quad \forall n \geq 0, \quad (6)$$

and  $f_\bullet$  is a fibration if

$$A_\bullet \longrightarrow \pi_0(A_\bullet) \times_{\pi_0(B_\bullet)} B_\bullet \quad (7)$$

is surjective.<sup>3</sup> We would like to transfer this homotopy theory to  $\mathbf{G}_{ft}$ .

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<sup>2</sup>The morphisms are defined in the standard way. Notice, that any morphism between local  $C^\infty$ -rings is automatically local.

<sup>3</sup>The right hand side consists of the union of connected components of  $B_\bullet$ , that intersect non-trivially the image of  $f_\bullet$ .

Since the subcategory  $\mathcal{G}_{ft}^{op} \subset SC^\infty\mathcal{R}^{op}$  is defined by putting conditions on homotopy types of objects, it is clear that if  $A_\bullet \in SC^\infty\mathcal{R}^{op}$  is weakly equivalent to  $B_\bullet \in \mathcal{G}_{ft}^{op}$ , then  $A_\bullet \in \mathcal{G}_{ft}^{op}$ . However, it is not true that  $\mathcal{G}_{ft}^{op}$  is a model category in its own right, since it does not have all finite colimits.

It is typical for geometric situations to have a model structure, but not to have all finite colimits. Such categories are called pseudo-model categories ([TV05]). Our definition is slightly more general than the one in [TV05], since in  $C^\infty$ -geometry it can happen that  $\Gamma(Sp(A), \mathcal{O}_{Sp(A)}) \not\cong A$ .

**Definition 4** *A right pseudo-model category is given by a model category  $\mathcal{M}$ , a full subcategory  $\mathcal{H} \subseteq \mathcal{M}$ , closed with respect to finite limits and weak equivalences in  $\mathcal{M}$ , and a full co-reflective subcategory  $\mathcal{P} \subseteq \mathcal{H}$ , with the right adjoint to the inclusion denoted by  $\mathcal{R} : \mathcal{H} \rightarrow \mathcal{P}$ , s.t.  $\mathcal{R}$  maps cofibrations and weak equivalences to cofibrations and weak equivalences respectively.<sup>4</sup>*

If we denote by  $\tilde{\mathcal{P}} \subseteq \mathcal{H}$  the full subcategory, consisting of objects that are weakly equivalent to objects in  $\mathcal{P}$ , we obviously obtain an adjunction

$$\subseteq : \mathcal{P} \rightleftarrows \tilde{\mathcal{P}} : \mathcal{R} \quad (8)$$

whose unit is invertible, and whose counit consists of weak equivalences. Thus we see that the simplicial localization of  $\mathcal{P}$  sits inside the simplicial localization of  $\mathcal{M}$  as a full simplicial subcategory. Moreover, as the following proposition shows,  $\mathcal{P}$  inherits a model structure from  $\mathcal{M}$ .<sup>5</sup> The proof is straightforward.

**Proposition 3** *Let  $\mathcal{C}, \mathcal{F}, \mathcal{W} \subseteq \mathcal{M}$  be the subcategories of cofibrations, fibrations, and weak equivalences respectively. Then  $(\mathcal{C} \cap \mathcal{P}, \mathcal{W} \cap \mathcal{P})$  define a model structure on  $\mathcal{P}$ , with the fibrations being retracts of  $\mathcal{R}(\mathcal{F})$ .*

By definition  $\mathcal{G}_{ft}^{op} \subset SC^\infty\mathcal{R}^{op}$  is closed with respect to weak equivalences, and it is easy to check that it is closed with respect to all finite limits. The functor  $\mathbf{Sp} : \mathcal{G}_{ft}^{op} \rightarrow \mathbf{G}_{ft}$  has a left adjoint  $\Gamma$ , and from Proposition 2 we know that  $\mathbf{Sp}$  maps weak equivalences to weak equivalences, and the unit  $Id_{\mathbf{G}_{ft}} \rightarrow \mathbf{Sp} \circ \Gamma$  is an isomorphism.

Finally, it is straightforward to prove that given a fibration  $A_\bullet \rightarrow B_\bullet$  of simplicial  $C^\infty$ -rings, if we invert every  $a \in A_0$ ,  $b \in B_0$ , that do not vanish on

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<sup>4</sup>A morphism in  $\mathcal{P}$  is a weak equivalence/cofibration, if it is a weak equivalence/cofibration as a morphism in  $\mathcal{H}$ .

<sup>5</sup>As in [Ho99], we distinguish between a model structure on a category, and a model category, the latter being a finitely complete and cocomplete category, together with a model structure.

$Sp(\pi_0(A_\bullet))$ ,  $Sp(\pi_0(B_\bullet))$  respectively, we get a fibration again. This means that  $\mathbf{Sp}$  maps cofibrations to cofibrations. Altogether we have the following statement.

**Proposition 4** *The quadruple  $(SC^\infty\mathcal{R}^{op}, \mathcal{G}_{ft}^{op}, \mathbf{G}_{ft}, \mathbf{Sp})$  is a right pseudo-model category.*

The advantage of working with the model structure on  $\mathbf{G}_{ft}$ , instead of  $SC^\infty\mathcal{R}$ , is that the notion of fibration in  $\mathbf{G}_{ft}$  is better adapted to geometry. Indeed, as the following proposition shows, starting with a fibrant scheme and restricting to a locally closed subset, we get a fibrant scheme again. The proof is straightforward.

**Proposition 5** *1. Let  $(X, \mathcal{O}_{\bullet, X}) \in \mathbf{G}_{ft}$ , and let  $\iota : Y \subseteq X$  be a locally closed subset. Then  $(Y, \iota^{-1}(\mathcal{O}_{\bullet, X})) \in \mathbf{G}_{ft}$ .*  
*2. If  $(X, \mathcal{O}_{\bullet, X})$  is fibrant, so is  $(Y, \iota^{-1}(\mathcal{O}_{\bullet, X}))$ .*

An immediate consequence of Proposition 5, is that every manifold  $\mathcal{M}$  of finite type<sup>6</sup> is fibrant, as an object of  $\mathbf{G}_{ft}$ . Indeed, such  $\mathcal{M}$  is embeddable into some  $\mathbb{R}^n$ , and then  $\mathcal{M}$  is a retract of its tubular neighbourhood, which is fibrant, since it is an open subscheme of  $\mathbb{R}^n$ .

Another example of a fibration is the trivial bundle  $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ , since it is the spectrum of the free  $C^\infty$ -morphism  $C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n+m})$ . Consequently, a trivial bundle over any manifold  $\mathcal{M}$  of finite type is a fibration, since it is a retract of a trivial bundle over some  $\mathbb{R}^n$ . Then any vector bundle over  $\mathcal{M}$  is a fibration, since any vector bundle is a retract of a trivial one.

We finish this subsection by noting that any inclusion of a closed cosimplicial  $C^\infty$ -subscheme is a cofibration, since it corresponds to a surjective morphism of simplicial  $C^\infty$ -rings. In particular, every  $\mathbf{X} \in \mathbf{G}_{ft}$  is cofibrant. Note however, that there are cofibrations in  $\mathbf{G}_{ft}$ , that are not inclusions of closed subschemes. For example any smooth morphism between manifolds is a cofibration.

## 2.2 Enrichment in $S\mathcal{S}et$

The category  $SC^\infty\mathcal{R}$  of simplicial  $C^\infty$ -rings is a simplicial model category. The simplicial structure is defined as follows ([Qu67], section II.1): for any

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<sup>6</sup>Meaning a manifold  $\mathcal{M}$ , s.t.  $C^\infty(\mathcal{M})$  is a finitely generated  $C^\infty$ -ring. This is equivalent to being able to embed  $\mathcal{M}$  into  $\mathbb{R}^n$  for some finite  $n$ .

$A_\bullet \in SC^\infty\mathcal{R}$ , and any  $K \in SSet$  we have

$$(A_\bullet \otimes K)_n := \coprod_{k_n \in K_n} A_n, \quad (9)$$

where the coproduct is taken in the category  $C^\infty\mathcal{R}$  of  $C^\infty$ -rings. For any weakly order preserving map  $f : m \rightarrow n$  in  $\Delta$  we have

$$f^* : (A_\bullet \otimes K)_n \rightarrow (A_\bullet \otimes K)_m, \quad f^*(a_{k_n}) := (f^*(a))_{f^*(k_n)}, \quad (10)$$

where  $a \in A_n$ , and  $a_{k_n}$  belongs to the copy of  $A_n$ , that is indexed by  $k_n \in K_n$ .

This gives an enrichment of  $SC^\infty\mathcal{R}$  in  $SSet$  by adjunction:

$$\underline{Hom}(A_\bullet, B_\bullet) := \{Hom(A_\bullet \otimes \Delta[k], B_\bullet)\}_{k \geq 0}, \quad (11)$$

and there is the second left adjoint to  $\underline{Hom}(-, -)$ :

$$\underline{Hom}(A_\bullet, B_\bullet) \cong \{Hom(A_\bullet, B_\bullet^{\Delta[k]})\}_{k \geq 0}. \quad (12)$$

The category  $\mathbf{G}_{ft}$  does not inherit a simplicial structure from  $SC^\infty\mathcal{R}^{op}$ . However, it does inherit a part of it, which is enough for doing homotopical computations. First we give such categories a name. We will say that a simplicial set  $S_\bullet$  is of finite type, if  $\pi_0(S_\bullet)$  is finite. Let  $SSet_{ft} \subset SSet$  be the full subcategory, consisting of simplicial sets of finite type.

**Definition 5** A right simplicial category is a category  $\mathcal{P}$ , together with bifunctors

$$\underline{Hom}(-, -) : \mathcal{P}^{op} \times \mathcal{P} \longrightarrow SSet, \quad (13)$$

$$-^- : \mathcal{P}^{op} \times SSet_{ft} \longrightarrow \mathcal{P}, \quad (14)$$

s.t.  $\forall K, L \in SSet_{ft}, \forall A, B \in \mathcal{P}$  there are natural coherent isomorphisms

$$A^{pt} \cong A, (A^K)^L \cong A^{K \times L}, Hom(A, B^K) \cong Hom(K, \underline{Hom}(A, B)). \quad (15)$$

Just as with the usual simplicial categories,  $\underline{Hom}(-, -)$  makes  $\mathcal{P}$  into a  $SSet$ -category, and  $Hom_{\mathcal{P}}(-, -) \cong \underline{Hom}(-, -)_0$ . Similar to the full simplicial case, there is a notion of compatibility between the structures of a right simplicial category and a right pseudo-model category, as follows.

**Definition 6** A right simplicial pseudo-model category consists of

- a simplicial model category  $\mathcal{M}$ ,



- a right pseudo-model category  $(\mathcal{M}, \mathcal{H}, \mathcal{P}, \mathcal{R})$ , extending the model structure on  $\mathcal{M}$ , and such that  $\forall K \in SSet_{ft}, \forall A \in \mathcal{H}$

$$A^K \in \mathcal{H}, \quad (16)$$

- a right simplicial structure on  $\mathcal{P}$ , s.t.  $\forall A \in \mathcal{H}, \forall K \in SSet_{ft}$  there is a coherent natural isomorphism

$$\mathcal{R}(A^K) \cong (\mathcal{R}(A))^K. \quad (17)$$

Just as in the full simplicial case, a right simplicial structure on a right pseudo-model category lets us calculate mapping spaces. For  $A, B \in \mathcal{P}$  we define

$$\underline{Hom}(A, B) := \{Hom_{\mathcal{P}}(A, B^{\Delta[n]})\}_{n \geq 0}. \quad (18)$$

The following proposition shows that these mapping spaces behave as expected. The proof is standard.

**Proposition 6** 1. Let  $(\mathcal{M}, \mathcal{H}, \mathcal{P}, \mathcal{R})$  be a right simplicial pseudo-model category. Let  $j : A \rightarrow B$ ,  $q : X \rightarrow Y$  be a cofibration and a fibration in  $\mathcal{P}$ , respectively. Then

$$j * q : \underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X) \times_{\underline{Hom}(A, Y)} \underline{Hom}(B, Y) \quad (19)$$

is a fibration of simplicial sets, which is trivial if  $j$  or  $q$  is trivial.

2. Let  $A, X \in \mathcal{P}$ , with  $A$  being cofibrant and  $X$  being fibrant. Then  $\underline{Hom}(A, X)$  is a Kan complex, and it is weakly equivalent to the corresponding mapping space in the simplicial localization of  $\mathcal{P}$ .

We would like to define a right simplicial structure on  $\mathbf{G}_{ft}$ , s.t. it is compatible with the right pseudo-model structure from Proposition 4. In fact there is a natural right simplicial structure on all of  $\mathbf{C}^\infty \mathbf{Sch}$ , defined as follows:  $\forall K \in SSet_{ft}, \forall (X, \mathcal{O}_{\bullet, X}) \in \mathbf{C}^\infty \mathbf{Sch}$  let

$$(X, \mathcal{O}_{\bullet, X})^K := \prod_{i \in \pi_0(K)} (X, \mathcal{O}_{\bullet, X} \otimes K_i), \quad (20)$$

where  $K_i \subseteq K$  is the connected component corresponding to  $i \in \pi_0(K)$ ,  $\mathcal{O}_{\bullet, X} \otimes K_i$  is the sheaf of simplicial  $C^\infty$ -rings, generated by

$$U \longmapsto \Gamma(U, \mathcal{O}_{\bullet, X}) \otimes K_i, \quad (21)$$

and the product is taken in  $\mathbf{C}^\infty \mathbf{Sch}$ . We claim that  $(X, \mathcal{O}_{\bullet, X})^K \in \mathbf{C}^\infty \mathbf{Sch}$ . This is an immediate consequence of the following lemma.

**Lemma 1** *For any  $A_\bullet \in SC^\infty\mathcal{R}$ , any  $K \in SSet_{ft}$ , s.t.  $\pi_0(K) = pt$ , there is a natural isomorphism*

$$\mathbf{Sp}(A_\bullet \otimes K) \longrightarrow \mathbf{Sp}(A_\bullet)^K. \quad (22)$$

**Proof:** Consider  $A_\bullet \otimes K \rightarrow A_\bullet$ , given by the unique  $K \rightarrow \Delta[0]$ . It is easy to see that this map induces an isomorphism  $\pi_0(A_\bullet \otimes K) \rightarrow \pi_0(A_\bullet)$ . Therefore, we have a chosen homeomorphism

$$Sp(A_\bullet) \xrightarrow{\cong} Sp(A_\bullet \otimes K). \quad (23)$$

Clearly, it is natural in  $A_\bullet$  and  $K$ , and we will use it to identify  $Sp(A_\bullet)$  and  $Sp(A_\bullet \otimes K)$ . We will denote both of them by  $Sp(A_\bullet)$ .

Let  $O$  be the following pre-sheaf of simplicial  $C^\infty$ -rings on  $Sp(A_\bullet)$ :

$$U \longmapsto \{A_n\{(s^n\mathcal{U})^{-1}\}\}_{n \geq 0}, \quad (24)$$

where  $\mathcal{U} \subset A_0$  consists of elements, that do not vanish on  $U$ . Also let  $O'$  be the pre-sheaf

$$U \longmapsto \{(A_\bullet \otimes K)_n\{(s^n\mathcal{U}')^{-1}\}\}_{n \geq 0}, \quad (25)$$

where  $\mathcal{U}' \subset (A_\bullet \otimes K)_0$  consists of the elements, that do not vanish on  $U$ . We claim that there is a natural morphism of pre-sheaves

$$O \otimes K \longrightarrow O'. \quad (26)$$

Indeed, let  $k \in K_n$ , let  $\iota_k : A_n \rightarrow \coprod_{K_n} A_n$  be the corresponding inclusion, and let  $\tilde{\iota}_k : A_n \rightarrow (\coprod_{K_n} A_n)\{(s^n\mathcal{U}')^{-1}\}$  be the composition with localization. It is obvious that  $\tilde{\iota}_k$  inverts every element of  $s^n(\mathcal{U})$ , and hence we have

$$\coprod_{K_n} (A_n\{(s^n\mathcal{U})^{-1}\}) \longrightarrow (\coprod_{K_n} A_n)\{(s^n\mathcal{U})^{-1}\}, \quad (27)$$

giving us (26). Sheafification of (26) produces a morphism of sheaves

$$\mathcal{O}_{\bullet, Sp(A_\bullet)} \otimes K \longrightarrow \mathcal{O}_{\bullet, A_\bullet \otimes K}. \quad (28)$$

It is straightforward to see that this morphism is an isomorphism on stalks. ■

Since  $\mathbf{Sp}$  is a right adjoint, it is clear that (22) is an isomorphism for any  $K \in SSet_{ft}$ . It is also easy to see that  $(X, \mathcal{O}_{\bullet, X})^K \in \mathbf{G}_{ft}$ , for any  $(X, \mathcal{O}_{\bullet, X}) \in \mathbf{G}_{ft}$ , and  $K \in SSet_{ft}$ . Thus we have proved the following proposition.

**Proposition 7** *The right pseudo-model category  $(SC^\infty\mathcal{R}^{op}, \mathcal{G}_{ft}^{op}, \mathbf{G}_{ft}, \mathbf{Sp})$  is right simplicial, with the simplicial structure given by (20).*

### 2.3 Derived manifolds

Using the weak equivalences in  $\mathbf{G}_{ft}$ , we have the notion of homotopy limits,<sup>7</sup> and since  $\mathbf{G}_{ft}$  is a right simplicial pseudo-model category, these homotopy limits are computable. Here we compute a particular kind of homotopy limit, that is essential for the notion of derived manifolds.

**Definition 7** • A Kuranishi neighbourhood is a homotopy limit in  $\mathbf{G}_{ft}$  of a diagram

$$\mathbb{R}^n \begin{array}{c} \xrightarrow{\omega} \\ \xrightarrow{\sigma} \end{array} \mathbb{R}^{m+n} \quad (29)$$

where  $\omega$  is the 0-section, and  $\sigma$  is any smooth section.

- A derived manifold of finite type is any  $\mathbf{X} \in \mathbf{G}_{ft}$ , s.t. locally it is weakly equivalent to Kuranishi neighbourhoods.
- We will denote the full simplicial subcategory of  $\mathbf{G}_{ft}$ , consisting of derived manifolds, by  $\mathbf{Man}_{ft}$ .

We would like to compute the homotopy limit of (29) explicitly. The right simplicial pseudo-model structure on  $\mathbf{G}_{ft}$  allows us to do this. In fact, the same approach works in a more general situation.

Let  $\mathcal{M}$  be a manifold of finite type,<sup>8</sup> and let  $\xi : E \rightarrow \mathcal{M}$  be a vector bundle over  $\mathcal{M}$ . Let  $\sigma : \mathcal{M} \rightarrow E$  be any smooth section, and let  $\omega : \mathcal{M} \rightarrow E$  be the 0-section. We would like to compute the homotopy equalizer in the following diagram:

$$\mathcal{M} \begin{array}{c} \xrightarrow{\omega} \\ \xrightarrow{\sigma} \end{array} E. \quad (30)$$

Since  $\omega, \sigma$  are sections of  $\xi$ , computing the usual equalizer of (30) is equivalent to computing the pullback of

$$\begin{array}{ccc} & \mathcal{M} & \\ & \downarrow \omega & \\ \mathcal{M} & \xrightarrow{\sigma} & E \end{array} \quad (31)$$

in the category  $\mathbf{G}_{ft}/\mathcal{M}$ . Comparing fibrant replacements of (30) and (31), we see that also homotopy limits of (30), (31) agree.

We know (Proposition 5, and the discussion immediately after) that  $\xi : E \rightarrow \mathcal{M}$  is a fibration, and hence  $E$  is fibrant in  $\mathbf{G}_{ft}/\mathcal{M}$ . Therefore,

<sup>7</sup> They are defined as limits in the simplicial localization of  $\mathbf{G}_{ft}$ .

<sup>8</sup> Recall that  $\mathcal{M}$  is of finite type, if  $C^\infty(\mathcal{M})$  is a finitely generated  $C^\infty$ -ring.

to compute the homotopy pullback of (31) it is enough to find a fibrant resolution of only one of  $\omega$ ,  $\sigma$  (e.g. [Lu09], proposition A.2.4.4). Now we are going to produce a resolution of  $\sigma$ .

Let  $i_0, i_1 : \Delta[0] \rightrightarrows \Delta[1]$  be the canonical inclusions. They are trivial cofibrations, and hence, since  $E$  is fibrant, the morphism  $i_1^* : E^{\Delta[1]} \rightarrow E$  is a trivial fibration. Consider the following pullback diagram in  $\mathbf{G}_{ft}$ :

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{\tau} & E^{\Delta[1]} \\ j \downarrow & & \downarrow i_1^* \\ \mathcal{M} & \xrightarrow{\sigma} & E \end{array} \quad (32)$$

Obviously  $j$  is a trivial fibration, and it has a section ([GJ99], proof of Lemma II.8.4). Indeed, consider the following diagram

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\sigma} & E & \xrightarrow{\delta^*} & E^{\Delta[1]} \\ \downarrow = & & & & \downarrow i_1^* \\ \mathcal{M} & \xrightarrow{\sigma} & E & & E \end{array} \quad (33)$$

where  $\delta : \Delta[1] \rightarrow \Delta[0]$ . This diagram commutes because  $i_1$  is a section of  $\delta$ . Therefore there is a unique  $l : \mathcal{M} \rightarrow \mathcal{M}'$ , s.t.

$$j \circ l = Id_{\mathcal{M}}, \quad \tau \circ l = \delta^* \circ \sigma. \quad (34)$$

Moreover, since  $i_0$  is also a section of  $\delta$ , we have  $i_0^* \circ \tau \circ l = \sigma$ , i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\sigma} & E \\ l \downarrow & & \downarrow = \\ \mathcal{M}' & \xrightarrow{i_0^* \circ \tau} & E \end{array} \quad (35)$$

One can show (loc. cit.) that  $i_0^* \circ \tau$  is a fibration, and therefore  $i_0^* \circ \tau$  is a fibrant replacement of  $\sigma$  ( $l$  is a weak equivalence, since it is a section of a trivial fibration). So to compute the homotopy limit of (31) it is enough to compute the usual limit in  $\mathbf{G}_{ft}/\mathcal{M}$  of

$$\begin{array}{ccc} & & \mathcal{M} \\ & & \downarrow \omega \\ \mathcal{M}' & \xrightarrow{i_0^* \circ \tau} & E \end{array} \quad (36)$$

Now we would like to write (36) in terms of the  $C^\infty$ -rings of functions. Let  $\omega^*, \sigma^* : C^\infty(E) \rightarrow C^\infty(\mathcal{M})$  be the  $C^\infty$ -morphisms, corresponding to  $\omega, \sigma$ .

According to Proposition 7

$$E^{\Delta[1]} \cong \mathbf{Sp}(C^\infty(E) \otimes \Delta[1]). \quad (37)$$

For every  $k \geq 0$  the set  $\Delta[1]_k$  consists of weakly order preserving maps  $\eta : \underline{k} \rightarrow \underline{1}$ . There are  $k+2$  such maps, and we denote them by  $\{\eta_k^0, \dots, \eta_k^{k+1}\}$ , where  $\eta_k^i$  sends  $\{0, \dots, i-1\} \mapsto 0$  and  $\{i, \dots, k\} \mapsto 1$ . Therefore we have

$$(C^\infty(E) \otimes \Delta[1])_k = C^\infty(E)^0 \otimes_{\infty} \dots \otimes_{\infty} C^\infty(E)^{k+1}, \quad (38)$$

where  $C^\infty(E)^i$  is a copy of  $C^\infty(E)$ , corresponding to  $\eta_k^i$ ,  $0 \leq i \leq k+1$ . For every  $k \geq 0$  the set  $\Delta[0]_k$  consists of the unique map  $\zeta_k : \underline{k} \rightarrow \underline{0}$ , and by definition

$$i_0(\zeta_k) := \eta_k^0, \quad i_1(\zeta_k) := \eta_k^{k+1}. \quad (39)$$

Since  $\mathbf{Sp} : \mathcal{G}_{ft}^{op} \rightarrow \mathbf{G}_{ft}$  has a left adjoint, it preserves limits, and we have

$$\mathcal{M}' \cong \mathbf{Sp}(\{C^\infty(E)^0 \otimes_{\infty} \dots \otimes_{\infty} C^\infty(E)^k \otimes_{\infty} C^\infty(\mathcal{M})^{k+1}\}_{k \geq 0}). \quad (40)$$

Then the limit of (36) is

$$\mathbf{Sp}(\{C^\infty(\mathcal{M})^0 \otimes_{\infty} C^\infty(E)^1 \otimes_{\infty} \dots \otimes_{\infty} C^\infty(E)^k \otimes_{\infty} C^\infty(\mathcal{M})^{k+1}\}_{k \geq 0}), \quad (41)$$

where the simplicial structure maps are given by the structure maps on  $\Delta[1]$ , identities on  $C^\infty(E)$ , and  $\omega^*, \sigma^*$ ; with  $\omega^*$  being used when  $C^\infty(\mathcal{M})$  is indexed by  $\eta_k^0$ , and  $\sigma^*$  being used when  $C^\infty(\mathcal{M})$  is indexed by  $\eta_k^{k+1}$ .

We will denote  $\{C^\infty(\mathcal{M})^0 \otimes_{\infty} C^\infty(E)^1 \otimes_{\infty} \dots \otimes_{\infty} C^\infty(E)^k \otimes_{\infty} C^\infty(\mathcal{M})^{k+1}\}_{k \geq 0}$  by  $\mathcal{B}(\omega^*, \sigma^*)_{\bullet}$ , since it is clear that this is just the bar construction of  $C^\infty(E)$  with coefficients in  $C^\infty(\mathcal{M})$ .

The simplicial  $C^\infty$ -ring  $\mathcal{B}(\omega^*, \sigma^*)_{\bullet}$  is not very complicated, but it can be made even simpler, making homotopical calculations so much easier. Consider first the case when  $E = \mathcal{M} \times \mathbb{R}^m$  is a trivial bundle. Then we have  $C^\infty(E) = C^\infty(\mathcal{M}) \otimes_{\infty} C^\infty(\mathbb{R}^m)$ , and  $\omega^*, \sigma^*$  can be equivalently described using

$$C^\infty(\mathbb{R}^m) \xrightleftharpoons[\nu^*]{\mu^*} C^\infty(\mathcal{M}), \quad (42)$$

where  $\mu^*$  factors through  $\mathbb{R}$ , and  $\nu^*$  is some  $C^\infty$ -morphism. Let  $\mathcal{B}(\mu^*, \nu^*)_{\bullet}$  be the bar construction of  $C^\infty(\mathbb{R}^m)$  with coefficients in  $\mathbb{R}$ ,  $C^\infty(\mathcal{M})$ , i.e.

$$\mathcal{B}(\mu^*, \nu^*)_k = C^\infty(\mathbb{R}^m)^1 \otimes_{\infty} \dots \otimes_{\infty} C^\infty(\mathbb{R}^m)^k \otimes_{\infty} C^\infty(\mathcal{M})^{k+1}. \quad (43)$$

Now it is easy to see that  $\mathcal{B}(\omega^*, \sigma^*)_\bullet$  is a colimit of the following diagram

$$\begin{array}{ccc} C^\infty(\mathcal{M}) & \xrightarrow{(i_1)_*} & C^\infty(\mathcal{M}) \otimes \Delta[1] \\ \downarrow & & \\ \mathcal{B}(\mu^*, \nu^*)_\bullet & & \end{array} \quad (44)$$

Applying  $\mathbf{Sp}$  we obtain  $\mathbf{Sp}(\mathcal{B}(\omega^*, \sigma^*)_\bullet)$  as a limit of the following diagram

$$\begin{array}{ccc} & & \mathcal{M}^{\Delta[1]} \\ & & \downarrow i_1^* \\ \mathbf{Sp}(\mathcal{B}(\mu^*, \nu^*)_\bullet) & \longrightarrow & \mathcal{M} \end{array} \quad (45)$$

Therefore the morphism  $\mathbf{Sp}(\mathcal{B}(\omega^*, \sigma^*)_\bullet) \rightarrow \mathbf{Sp}(\mathcal{B}(\mu^*, \nu^*)_\bullet)$  is a weak equivalence. Now we notice that

$$\mathcal{B}(\mu^*, \nu^*)_\bullet \cong \{C^\infty(E^{\times k}_{\mathcal{M}})\}_{k \geq 0}, \quad (46)$$

and since every bundle is locally trivial, we obtain the following result.

**Proposition 8** *Let  $E \rightarrow \mathcal{M}$  be a vector bundle over a manifold  $\mathcal{M}$  of finite type, and let  $\sigma : \mathcal{M} \rightarrow E$  be a section. Let  $\omega : \mathcal{M} \rightarrow E$  be the 0-section. The homotopy equalizer of*

$$\mathcal{M} \begin{array}{c} \xrightarrow{\omega} \\ \xrightarrow{\sigma} \end{array} E \quad (47)$$

can be written as

$$\mathbf{Sp}(\{C^\infty(E^{\times k}_{\mathcal{M}})\}_{k \geq 0}). \quad (48)$$

By construction, for each  $k \geq 2$ , the  $C^\infty$ -ring  $C^\infty(E^{\times k}_{\mathcal{M}})$  is obtained as a coproduct of degenerations of  $C^\infty(E)$ . In other words  $\{C^\infty(E^{\times k}_{\mathcal{M}})\}_{k \geq 0}$  is a 1-skeletal simplicial  $C^\infty$ -ring. Using this fact it is easy to show that  $\mathbf{Sp}(\{C^\infty(E^{\times k}_{\mathcal{M}})\}_{k \geq 0})$  is fibrant. Indeed

$$C^\infty(\mathcal{M}) \rightleftarrows C^\infty(E) \quad (49)$$

is a retract of some

$$C^\infty(\mathcal{M}) \rightleftarrows C^\infty(F) \quad (50)$$

where  $F \rightarrow \mathcal{M}$  is a trivial bundle. Since  $C^\infty(\mathcal{M}) \rightarrow C^\infty(F)$  is a free morphism, we see that  $\mathbf{Sp}(\{C^\infty(F^{\times k}_{\mathcal{M}})\}_{k \geq 0})$  is fibrant, and hence so is  $\mathbf{Sp}(\{C^\infty(E^{\times k}_{\mathcal{M}})\}_{k \geq 0})$ .

Going back to Kuranishi neighbourhoods, we would like to fix a particular model for such objects, i.e. a particular choice of the homotopy pullback. The preceding discussion suggests the following definition.

**Definition 8** *A standard Kuranishi neighbourhood is  $\mathbf{Sp}(\{C^\infty(\mathbb{R}^{n+km})\}_{k \geq 0})$ , with the simplicial structure defined as above.*

### 3 D-manifolds

#### 3.1 The 2-category of $d$ -spaces

We define a  $d$ -space to be a quadruple  $(X, \mathcal{O}'_X, \mathcal{E}_X, d)$ , where  $X$  is a second countable, Hausdorff space;  $\mathcal{O}'_X$  is a sheaf of  $C^\infty$ -rings on  $X$ ;  $\mathcal{E}_X$  is a sheaf of  $\mathcal{O}'_X$ -modules; and  $d : \mathcal{E}_X \rightarrow \mathcal{O}'_X$  is a morphism of  $\mathcal{O}'_X$ -modules, s.t.

1.  $\mathcal{O}_X := \mathcal{O}'_X / d\mathcal{E}_X$  is a soft sheaf, and its stalks are local  $C^\infty$ -rings,
2.  $\mathcal{O}_X$  is locally finitely generated (as a sheaf of  $C^\infty$ -rings),
3.  $d\mathcal{E}_X \subset \mathcal{O}'_X$  is a sheaf of square-zero ideals, and

$$(d\mathcal{E}_X)\mathcal{E}_X = 0. \quad (51)$$

It is easy to see that this definition of a  $d$ -space is equivalent to Definition 4.1.4 in [Jo12], in the sense that they define the same objects. We will denote a  $d$ -space  $(X, \mathcal{O}'_X, \mathcal{E}_X, d)$  by  $\underline{X}$ .

A 1-morphism of  $d$ -spaces  $\underline{X} \rightarrow \underline{Y}$  is given by a triple  $\underline{\phi} := (\phi, \phi', \phi'')$ , where  $\phi : X \rightarrow Y$  is a continuous map,  $\phi' : \mathcal{O}'_Y \rightarrow \phi_*(\mathcal{O}'_X)$  is a morphism of sheaves of  $C^\infty$ -rings, and  $\phi'' : \mathcal{E}_Y \rightarrow \phi_*(\mathcal{E}_X)$  is a morphism of  $\mathcal{O}'_Y$ -modules, s.t. the following diagram is commutative

$$\begin{array}{ccc} \mathcal{E}_Y & \xrightarrow{\phi''} & \phi_*(\mathcal{E}_X) \\ d \downarrow & & \downarrow \phi_*(d) \\ \mathcal{O}'_Y & \xrightarrow{\phi'} & \phi_*(\mathcal{O}'_X) \end{array} \quad (52)$$

Given two 1-morphisms  $\underline{\phi}_1 : \underline{X} \rightarrow \underline{Y}$ ,  $\underline{\phi}_2 : \underline{Y} \rightarrow \underline{Z}$ , their composition is

$$\underline{\phi}_2 \circ \underline{\phi}_1 := (\phi_2 \circ \phi_1, (\phi_2)_*(\phi'_1) \circ \phi'_2, (\phi_2)_*(\phi''_1) \circ \phi''_2). \quad (53)$$

Clearly, commutativity of (52) implies that  $\phi'$  induces a morphism

$$\phi^\# : \mathcal{O}_Y = \mathcal{O}'_Y / d\mathcal{E}_Y \longrightarrow \phi_*(\mathcal{O}'_X) / \phi_*(d)(\phi_*(\mathcal{E}_X)) \cong \phi_*(\mathcal{O}_X). \quad (54)$$

We will say that  $\underline{\phi}, \underline{\psi} : \underline{X} \rightarrow \underline{Y}$  are scheme-theoretically equal if  $\phi = \psi$  and  $\phi^\sharp = \psi^\sharp$ . In this case the image of  $\psi' - \phi'$  lies in  $\phi_*(d\mathcal{E}_X)$ , which is a square-zero ideal, hence we have a  $C^\infty$ -derivation

$$\psi' - \phi' : \mathcal{O}'_Y \longrightarrow \phi_*(d\mathcal{E}_X), \quad (55)$$

and we define a 2-morphism from  $\underline{\phi}$  to  $\underline{\psi}$  to be a  $C^\infty$ -derivation<sup>9</sup>

$$\eta : \mathcal{O}'_Y \rightarrow \phi_*(\mathcal{E}_X), \quad (56)$$

lifting  $\psi' - \phi'$  and  $\psi'' - \phi''$ , i.e. making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{E}_Y & \xrightarrow{\psi'' - \phi''} & \phi_*(\mathcal{E}_X) \\ d \downarrow & \nearrow \eta & \downarrow \phi_*(d) \\ \mathcal{O}'_Y & \xrightarrow{\psi' - \phi'} & \phi_*(d\mathcal{E}_X) \end{array} \quad (57)$$

Consider a diagram of 1- and 2-morphisms:

$$\begin{array}{ccccc} \underline{X} & \xrightarrow{\underline{\phi}_1} & \underline{Y} & \xrightarrow{\underline{\phi}_2} & \underline{Z} \\ \parallel & & \parallel & & \parallel \\ \underline{X} & \xrightarrow{\underline{\chi}_1} & \underline{Y} & \xrightarrow{\underline{\chi}_2} & \underline{Z} \\ \parallel & & \parallel & & \parallel \\ \underline{X} & \xrightarrow{\underline{\psi}_1} & \underline{Y} & \xrightarrow{\underline{\psi}_2} & \underline{Z} \end{array} \quad \begin{array}{c} \downarrow \eta_1 \\ \downarrow \theta_1 \end{array} \quad \begin{array}{c} \downarrow \eta_2 \\ \downarrow \theta_2 \end{array} \quad (58)$$

It is immediate from (57), that  $\eta_1 + \theta_1$  is a 2-morphism from  $\underline{\phi}_1$  to  $\underline{\psi}_1$ . Thus we define the vertical composition as follows:

$$\theta_1 \circ \eta_1 := \eta_1 + \theta_1, \quad \theta_2 \circ \eta_2 := \eta_2 + \theta_2. \quad (59)$$

It is easy to see that  $(\phi_2)_*(\eta_1) \circ \phi'_2$  and  $(\phi_2)_*(\chi''_1) \circ \eta_2$  are 2-morphisms from  $\underline{\phi}_2 \circ \underline{\phi}_1$  to  $\underline{\phi}_2 \circ \underline{\chi}_1$  and from  $\underline{\phi}_2 \circ \underline{\chi}_1$  to  $\underline{\chi}_2 \circ \underline{\chi}_1$  respectively. Therefore, we define the horizontal composition as follows:

$$\eta_2 \square \eta_1 := (\phi_2)_*(\eta_1) \circ \phi'_2 + (\phi_2)_*(\chi''_1) \circ \eta_2, \quad (60)$$

$$\theta_2 \square \theta_1 := (\chi_2)_*(\theta_1) \circ \chi'_2 + (\chi_2)_*(\psi''_1) \circ \theta_2.$$

---

<sup>9</sup>Note that  $\phi^\sharp = \psi^\sharp$  implies that  $\phi', \psi'$  define the same  $\mathcal{O}'_Y$ -module structure on  $\phi_*(\mathcal{E}_X)$ .



It is straightforward to check, that  $\square$  is associative and unital, and the interchange condition  $(\theta_2 \square \theta_1) \circ (\eta_2 \square \eta_1) = (\theta_2 \circ \eta_2) \square (\theta_1 \circ \eta_1)$  is satisfied. Thus  $d$ -spaces, 1- and 2-morphisms form a strict 2-category, that we will denote by  $\underline{G}$ .

It is easy to check that  $\underline{G}$  is equivalent to the 2-category of  $d$ -spaces, defined in [Jo12]. The difference is only in the presentation: in [Jo12] derivations are written as maps out of sheaves of differential forms, and instead of pushforwards, one uses pullbacks of sheaves.

We will denote by  $\underline{G}_{ft} \subset \underline{G}$  the full 2-subcategory, consisting of  $d$ -spaces of finite type, i.e. those  $(X, \mathcal{O}'_X, \mathcal{E}_X, d)$  s.t.  $\mathcal{O}_X := \mathcal{O}'_X/d\mathcal{E}_X$  is a sheaf of finitely generated  $C^\infty$ -rings.

### 3.2 Truncation of cosimplicial $C^\infty$ -schemes

Let  $\mathbf{X} := (X, \mathcal{O}_{\bullet, X}) \in \mathbf{G}_{ft}$ . We would like to define a  $d$ -space, that is the truncation of  $\mathbf{X}$ . First we need to recall some standard notation from the theory of simplicial modules: let  $d_i^k : \mathcal{O}_{k, X} \rightarrow \mathcal{O}_{k-1, X}$  be the boundaries, one defines

$$\forall k \geq 1, \quad \mathcal{N}_k := \bigcap_{0 \leq i \leq k-1} \text{Ker}(d_i^k) \subset \mathcal{O}_{k, X}. \quad (61)$$

Clearly, this is a sequence of sheaves of ideals. One has  $d_{k-1}^{k-1} \circ d_k^k = 0$  on  $\mathcal{N}_*$ , and hence  $(\mathcal{N}_k, (-1)^k d_k^k)$  is a complex, called the normalized complex.

Now we define two sheaves on  $X$ :

$$\mathcal{O}'_X := \mathcal{O}_{0, X} / (d_1^1(\mathcal{N}_1))^2, \quad \mathcal{E} := \mathcal{N}_1 / (d_2^2(\mathcal{N}_2) + \mathcal{N}_1^2). \quad (62)$$

Since  $d_1^1 \circ d_2^2 = 0$  on  $\mathcal{N}_*$ , it is clear that  $d_1^1$  induces a morphism of  $\mathcal{O}'_X$ -modules

$$d : \mathcal{E} \rightarrow \mathcal{O}'_X. \quad (63)$$

**Proposition 9** *Defined as above  $(X, \mathcal{O}'_X, \mathcal{E}, d)$  is a  $d$ -space, and the assignment  $\mathbf{X} \mapsto (X, \mathcal{O}'_X, \mathcal{E}, d)$  extends to a functor  $\mathbf{T} : \mathbf{G}_{ft} \rightarrow \underline{G}_{ft}$ .*

**Proof:** Since cohomology of a normalized complex is isomorphic to the sequence of homotopy groups of the original simplicial module, it is clear that  $\mathcal{O}'_X/d\mathcal{E} \cong \mathcal{O}_X := \pi_0(\mathcal{O}_{\bullet, X})$ . Therefore, to prove that  $(X, \mathcal{O}'_X, \mathcal{E}, d)$  is a  $d$ -space, it is enough to show that

$$(d\mathcal{E})\mathcal{E} = 0. \quad (64)$$

Let  $U \subseteq X$  be open, and let  $a_1, a_2 \in \Gamma(U, \mathcal{N}_1)$ . Then

$$s_0(a_1)s_1(a_2) - s_1(a_1a_2) \in \Gamma(U, \mathcal{N}_2),^{10} \quad (65)$$

and clearly

$$d_2^2(s_0(a_1)s_1(a_2) - s_1(a_1a_2)) = d_1^1(a_1)a_2 - a_1a_2, \quad (66)$$

i.e. the class of  $d_1^1(a_1)a_2$  in  $\mathcal{E}$  is 0.

From functoriality of the normalized complex, it is clear that a morphism  $\mathbf{X} \rightarrow \mathbf{Y}$  induces a 1-morphism  $\mathbf{T}(\mathbf{X}) \rightarrow \mathbf{T}(\mathbf{Y})$ , and this assignment is functorial. ■

From the simplicial enrichment of  $\mathbf{G}_{ft}$  we can obtain a 2-category as follows. Let  $\mathbf{G}_{ft}^f \subset \mathbf{G}_{ft}$  be the full subcategory, consisting of fibrant schemes. Then, since every object in  $\mathbf{G}_{ft}$  is cofibrant, for any  $\mathbf{X}, \mathbf{Y} \in \mathbf{G}_{ft}^f$  the simplicial set  $\underline{Hom}(\mathbf{X}, \mathbf{Y})$  is fibrant.

Let  $\underline{\mathbf{G}}_{ft}$  be the 2-category consisting of the same objects and morphisms as  $\mathbf{G}_{ft}^f$ , and with 2-morphisms being homotopy classes of 1-simplices in mapping spaces of  $\mathbf{G}_{ft}^f$ . Clearly  $\underline{\mathbf{G}}_{ft}$  is a 2-category, where each 2-morphism is invertible.

**Proposition 10** *The truncation functor  $\mathbf{T}$ , defined above, extends to a 2-functor*

$$\underline{\mathbf{G}}_{ft} \longrightarrow \underline{\mathbf{G}}_{ft}. \quad (67)$$

**Proof:** Let  $\alpha^0, \alpha^1 : \mathbf{X} \rightarrow \mathbf{Y}$  be two 1-morphisms in  $\underline{\mathbf{G}}_{ft}$ , and let  $\beta : \mathbf{X} \rightarrow \mathbf{Y}^{\Delta[1]}$  be a 2-morphism from  $\alpha^0$  to  $\alpha^1$ . This means that  $i_0, i_1 : \Delta[0] \rightarrow \Delta[1]$  define a commutative diagram

$$\begin{array}{ccccc} & & \mathbf{X} & & \\ & \swarrow \alpha^0 & \downarrow \beta & \searrow \alpha^1 & \\ \mathbf{Y} & \xleftarrow{i_0^*} & \mathbf{Y}^{\Delta[1]} & \xrightarrow{i_1^*} & \mathbf{Y} \end{array} \quad (68)$$

By definition  $\mathbf{Y}^{\Delta[1]} = (Y, \mathcal{O}_{\bullet, Y} \otimes \Delta[1])$ , and the first two levels of  $\mathcal{O}_{\bullet, Y} \otimes \Delta[1]$  are

$$\mathcal{O}_{0, Y}^0 \amalg \mathcal{O}_{0, Y}^1, \quad \mathcal{O}_{1, Y}^0 \amalg \mathcal{O}_{1, Y}^1 \amalg \mathcal{O}_{1, Y}^2, \quad (69)$$

---

<sup>10</sup>Here  $s_0, s_1 : \mathcal{O}_{1, X} \rightarrow \mathcal{O}_{2, X}$  are the two degenerations.

where coproducts are taken in the category of sheaves of  $C^\infty$ -rings, and the superscript stands for indexing by simplices of  $\Delta[1]$ . Consider the following map

$$\nu : \mathcal{O}_{0,Y} \xrightarrow{s} \mathcal{O}_{1,Y}^1 \xrightarrow{\beta^\sharp} \beta_*(\mathcal{O}_{1,X}), \quad (70)$$

where  $s : \mathcal{O}_{0,Y} \rightarrow \mathcal{O}_{1,Y}$  is the degeneration. Recall that  $\mathcal{N}_1$  is a direct summand of  $\mathcal{O}_{1,X}$ , therefore, composing with the projection, we get

$$\mathcal{O}_{0,Y} \xrightarrow{\nu} \beta_*(\mathcal{O}_{1,X}) \longrightarrow \beta_*(\mathcal{N}_1) \longrightarrow \beta_*(\mathcal{N}_1/(d_2\mathcal{N}_2 + \mathcal{N}_1^2)). \quad (71)$$

Since in the second projection we divide by  $\mathcal{N}_1^2$ , it is clear that (71) factors through  $\mathcal{O}_{0,Y}/(d_1^1\mathcal{N}_1)^2$ , and hence we obtain

$$\eta : \mathcal{O}_{0,Y}/(d_1^1\mathcal{N}_1)^2 \longrightarrow \beta_*(\mathcal{N}_1/(d_2^2\mathcal{N}_2 + \mathcal{N}_1^2)). \quad (72)$$

It is straightforward to check that  $\eta$  is a 2-morphism from  $\mathbf{T}(\alpha^0)$  to  $\mathbf{T}(\alpha^1)$ , and moreover, the composition

$$Hom(\mathbf{X}, \mathbf{Y}^{\Delta[1]}) \times Hom(\mathbf{Y}, \mathbf{Z}^{\Delta[1]}) \longrightarrow Hom(\mathbf{X}, \mathbf{Z}^{\Delta[1]}), \quad (73)$$

given by the diagonal  $\Delta[1] \rightarrow \Delta[1] \times \Delta[1]$ , corresponds to the horizontal composition in  $\underline{\mathbf{G}}_{ft}$ . It is immediate to see that  $\mathbf{T}$  maps vertical composition in  $\mathbf{G}_{ft}$  to the vertical composition in  $\underline{\mathbf{G}}_{ft}$ . ■

Since we consider only fibrant cosimplicial  $C^\infty$ -schemes, a morphism in  $\mathbf{G}_{ft}^f$  is a weak equivalence, if and only if it has a quasi-inverse. Therefore, it is clear that  $\mathbf{T}$  maps weak equivalences to equivalences.

### 3.3 D-manifolds

Let  $\mathbf{X} \in \mathbf{G}_{ft}$  be a standard Kuranishi neighbourhood (Definition 8), i.e.  $\mathbf{X}$  is a homotopy pullback in  $\mathbf{G}_{ft}$  of

$$\begin{array}{ccc} & & \mathbb{R}^0 \\ & & \downarrow \\ \mathbb{R}^n & \xrightarrow{\nu} & \mathbb{R}^m \end{array} \quad (74)$$

We know that  $\mathbf{X} = \mathbf{Sp}(A_\bullet)$ , where  $A_\bullet$  is a 1-skeletal simplicial  $C^\infty$ -ring, with

$$A_0 = C^\infty(\mathbb{R}^n), \quad A_1 = C^\infty(\mathbb{R}^{n+m}). \quad (75)$$

Choosing coordinates  $\{x_1, \dots, x_n\}$ ,  $\{y_1, \dots, y_m\}$  on  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , we have that

$$\forall i, j, \quad d_0^1(y_i) := 0, \quad d_0^1(x_j) := x_j, \quad d_1^1(y_i) := f_i, \quad d_1^1(x_j) := x_j, \quad (76)$$

where  $f_i = \nu^*(y_i)$ . It is easy to see that

$$A_2 \supset \mathcal{N}_2 := \text{Ker}(d_0^2) \cap \text{Ker}(d_1^2) = \text{Ker}(d_0^2) \cdot \text{Ker}(d_1^2), \quad (77)$$

and hence

$$d_2^2(\mathcal{N}_2) = \sum_{1 \leq i, j \leq m} (y_i(y_j - f_j)). \quad (78)$$

Therefore

$$\begin{aligned} \mathcal{E} &:= \mathcal{N}_1 / (d_2^2(\mathcal{N}_2) + \mathcal{N}_1^2) = \\ &= \left( \sum_{1 \leq i \leq m} (y_i) / \sum_{1 \leq i, j \leq m} (y_i y_j) \right) \coprod_{C^\infty(\mathbb{R}^n)} C^\infty(\mathbb{R}^n) / \sum_{1 \leq i \leq m} (f_i). \end{aligned} \quad (79)$$

In other words,  $\mathcal{E}$  is obtained by taking the bundle of vertical forms on  $\xi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$  at the 0-section, and then restricting it to the subscheme  $\{\nu = 0\} \subseteq \mathbb{R}^n$ . Since fibers of  $\xi$  are linear spaces, the bundle of vertical vector fields at the 0-section is naturally identified with  $\xi$  itself, and hence  $\mathcal{E}$  is naturally isomorphic to the bundle, obtained by restricting  $\xi^*$  to  $\{\nu = 0\}$ .

Given an  $\mathbf{X}$  as above, we will call its truncation  $\mathbf{T}(\mathbf{X})$  a standard model of a  $d$ -manifold. This obviously agrees with [Jo12], Definition 5.13, and hence we define  $d$ -manifolds to be  $d$ -spaces, that are locally equivalent in  $\underline{G}$  to standard models. This definition is different from [Jo12] only in that we do not require equidimensionality. We denote by  $\underline{Man}_{ft} \subset \underline{G}_{ft}$  the full 2-subcategory, consisting of  $d$ -manifolds of finite type.

Recall that  $\mathbf{Man}_{ft} \subset \mathbf{G}_{ft}$  is the full simplicial subcategory, consisting of derived manifolds of finite type. Let  $\mathbf{Man}_{ft}^f \subset \mathbf{Man}_{ft}$  be the full simplicial subcategory, consisting of fibrant derived manifolds of finite type. Correspondingly, let  $\underline{\mathbf{Man}}_{ft} \subset \underline{\mathbf{G}}_{ft}$  be the full 2-subcategory, consisting of derived manifolds of finite type. Since  $\underline{\mathbf{Man}}_{ft} \subset \underline{\mathbf{G}}_{ft}$  consists of objects that are locally equivalent to Kuranishi neighbourhoods, it is clear that the 2-functor  $\mathbf{T}$  maps  $\underline{\mathbf{Man}}_{ft}$  to  $\underline{Man}_{ft}$ .

We would like to investigate the properties of

$$\mathbf{T} : \underline{\mathbf{Man}}_{ft} \longrightarrow \underline{Man}_{ft}, \quad (80)$$

i.e. to understand if  $\mathbf{T}$  is full, faithful, essentially surjective etc. We will not do this in full generality, but only when restricted to the full 2-subcategories

$\underline{\mathbf{Man}}_{st} \subset \underline{\mathbf{Man}}_{ft}$  and  $\underline{Man}_{st} \subset \underline{Man}_{ft}$ , consisting of derived manifolds and  $d$ -manifolds that are 0-loci of sections of vector bundles over manifolds of finite type. For example all compact derived and  $d$ -manifolds are of this kind ([Sp10] and [Jo12] Theorems 6.28 and 6.33).

First of all, it is clear that every  $d$ -manifold in  $\underline{Man}_{st}$  is equivalent to the truncation of a derived manifold in  $\underline{\mathbf{Man}}_{st}$ . In fact we can say more. Let  $\mathfrak{K}$  be the category defined as follows:

- objects are triples  $(\mathcal{M}, E, \sigma)$ , where  $\mathcal{M}$  is a manifold of finite type,  $E$  is a bundle over  $\mathcal{M}$ , and  $\sigma$  is a smooth section of  $E$ ;
- morphisms are pairs of smooth maps  $\alpha : \mathcal{M} \rightarrow \mathcal{M}'$ ,  $\beta : E \rightarrow E'$ , s.t. the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\beta} & E' \\ \begin{array}{c} \uparrow \downarrow \\ 0 \end{array} \sigma & & \begin{array}{c} \uparrow \downarrow \\ 0 \end{array} \sigma' \\ \mathcal{M} & \xrightarrow{\alpha} & \mathcal{M}' \end{array} \quad (81)$$

There are two functors  $\mathfrak{K} \rightarrow \underline{\mathbf{Man}}_{st}$  and  $\mathfrak{K} \rightarrow \underline{Man}_{st}$ . The former is given by taking the homotopy equalizer of  $\sigma$  with the 0-section, and the latter is given by restricting  $E^*$  to the 0-locus of  $\sigma$  ([Jo12], Definition 5.13). As we have seen at the beginning of this section, the truncation functor completes the following commutative diagram

$$\begin{array}{ccc} & \mathfrak{K} & \\ \swarrow & & \searrow \\ \underline{\mathbf{Man}}_{st} & \xrightarrow{\mathbf{T}} & \underline{Man}_{st} \end{array} \quad (82)$$

It follows immediately that the inclusion of the usual theory of manifolds into  $\underline{Man}_{ft}$  factors through  $\underline{\mathbf{Man}}_{ft}$ . Another immediate consequence is the following result.

**Proposition 11** *The truncation 2-functor  $\mathbf{T} : \underline{\mathbf{Man}}_{st} \rightarrow \underline{Man}_{st}$  induces a surjection between the sets of equivalence classes of objects.*

Next we investigate the question of fullness. First we consider the special case, when the bundle is trivial and the manifold is just  $\mathbb{R}^n$ .

**Lemma 2** *Let  $\mathbf{X}, \mathbf{Y} \in \mathbf{G}_{ft}$ , with  $\mathbf{Y}$  being a standard Kuranishi neighbourhood (Definition 8). Then the map*

$$\mathbf{T} : Hom_{\mathbf{G}_{ft}}(\mathbf{X}, \mathbf{Y}) \longrightarrow Hom_{\underline{\mathbf{G}}_{ft}}(\mathbf{T}(\mathbf{X}), \mathbf{T}(\mathbf{Y})) \quad (83)$$

*is surjective.*

**Proof:** By assumption  $\mathbf{Y} = \mathbf{Sp}(\mathcal{B}(\mu^*, \nu^*)_{\bullet})$ , where

$$\mu^*, \nu^* : C^\infty(\mathbb{R}^m) \longrightarrow C^\infty(\mathbb{R}^n), \quad (84)$$

with  $\mu^*$  being evaluation at the origin. Since  $\mathbf{Sp}$  is right adjoint to  $\Gamma$ , we have a natural bijection

$$Hom_{\mathbf{G}_{ft}}(\mathbf{X}, \mathbf{Y}) \cong Hom_{SC^\infty\mathcal{R}}(\mathcal{B}(\mu^*, \nu^*)_{\bullet}, \Gamma(X, \mathcal{O}_{\bullet, X})). \quad (85)$$

Let  $\{x_1, \dots, x_n\}$ ,  $\{y_1, \dots, y_m\}$  be the coordinates on  $\mathbb{R}^n$ ,  $\mathbb{R}^m$  respectively. Then  $\mathcal{B}(\mu^*, \nu^*)_{\bullet}$  is a 1-skeletal, almost free simplicial  $C^\infty$ -ring, with  $x, y$  being the almost free generators. This means that the set of morphisms  $\mathcal{B}(\mu^*, \nu^*)_{\bullet} \rightarrow \Gamma(X, \mathcal{O}_{\bullet, X})$  is in bijective correspondence with the set of assignments

$$x_i \longmapsto f_i \in \Gamma(X, \mathcal{O}_{0, X}), \quad y_j \longmapsto g_j \in \Gamma(X, \mathcal{O}_{1, X}), \quad (86)$$

such that

$$d_0^1(g_j) = 0, \quad d_1^1(g_j) = \nu_j^*(f_1, \dots, f_n), \quad (87)$$

where  $\nu_j^* := \nu^*(y_j) \in C^\infty(\mathbb{R}^n)$ .

Let  $\underline{\phi} \in Hom_{\underline{\mathbf{G}}_{ft}}(\mathbf{T}(\mathbf{X}), \mathbf{T}(\mathbf{Y}))$ . Since  $C^\infty(\mathbb{R}^n)$  is a free  $C^\infty$ -ring, we can find a  $\phi_0$ , making the following diagram commutative:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \longrightarrow & \Gamma(Y, \mathcal{O}_{0, Y} / (d_1^1 \mathcal{N}_1)^2) \\ \downarrow \phi_0 & & \downarrow \underline{\phi} \\ \Gamma(X, \mathcal{O}_{0, X}) & \longrightarrow & \Gamma(X, \mathcal{O}_{0, X} / (d_1^1 \mathcal{N}_1)^2) \end{array} \quad (88)$$

Similarly, we can find  $\tilde{\phi}_1$ , making the following diagram commutative:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^m) & \longrightarrow & \Gamma(Y, \mathcal{N}_1 / (d_2^2 \mathcal{N}_2 + \mathcal{N}_1^2)) \\ \downarrow \tilde{\phi}_1 & & \downarrow \underline{\phi} \\ \Gamma(X, \mathcal{N}_1) & \longrightarrow & \Gamma(X, \mathcal{N}_1 / (d_2^2 \mathcal{N}_2 + \mathcal{N}_1^2)) \end{array} \quad (89)$$

It might happen that  $d_1^1(\tilde{\phi}_1(y_j)) \neq \nu_j^*(\phi_0(x_1), \dots, \phi_0(x_n))$ , however, clearly

$$\nu_j^*(\phi_0(x_1), \dots, \phi_0(x_n)) - d_1^1(\tilde{\phi}_1(y_j)) \in (d_1^1 \mathcal{N}_1)^2. \quad (90)$$

Therefore, we can choose  $\epsilon_j \in \Gamma(X, \mathcal{N}_1^2)$ , s.t.

$$d_1^1(\tilde{\phi}_1(y_j) + \epsilon_j) = \nu_j^*(\phi_0(x_1), \dots, \phi_0(x_n)). \quad (91)$$

Thus, defining

$$x_i \longmapsto \phi_0(x_i), \quad y_j \longmapsto \tilde{\phi}_1(y_j) + \epsilon_j \quad (92)$$

we are done. ■

To extend Lemma 2 to the cases where the codomain is not standard Kuranishi, we will need to glue morphisms, using softness of the structure sheaves. The following result provides the means for this.

**Lemma 3** *Let  $(\phi, \phi^\sharp) : \mathbf{X} \rightarrow \mathbf{Y}$  be a morphism in  $\mathbf{G}_{ft}$ , and suppose that  $\mathbf{Y}$  is a standard Kuranishi neighbourhood. For each open  $U \subseteq Y$  let*

$$\mathcal{F}(U) := \{\psi^\sharp : \mathcal{O}_{\bullet, Y}|_U \rightarrow \phi_*(\mathcal{O}_{\bullet, X})|_U \text{ s.t. } \mathbf{T}(\phi|_U, \psi^\sharp) = \mathbf{T}(\phi, \phi^\sharp)|_U\}. \quad (93)$$

*Then  $U \mapsto \mathcal{F}(U)$  is a soft sheaf (of sets) on  $Y$ .*

**Proof:** By assumption  $\mathbf{Y} = \mathbf{Sp}(\mathcal{B}(\mu^*, \nu^*), \bullet)$ , where

$$\mu^*, \nu^* : C^\infty(\mathbb{R}^m) \longrightarrow C^\infty(\mathbb{R}^n), \quad (94)$$

with  $\mu^*$  being evaluation at the origin. Let  $x_1, \dots, x_n, y_1, \dots, y_m$  be the coordinates on  $\mathbb{R}^n, \mathbb{R}^m$ , and let  $\nu_j^* := \nu^*(y_j) \in C^\infty(\mathbb{R}^n)$ .

Let  $V \subseteq Y$  be closed, we denote by  $\{[x_i]_V\}, \{[y_j]_V\}$  the images of  $\{x_i\}, \{y_j\}$  in  $\Gamma(V, \mathcal{O}_{0, Y}), \Gamma(V, \mathcal{O}_{1, Y})$  respectively.

Let  $\psi^\sharp \in \mathcal{F}(V)$ , clearly

$$\psi^\sharp([x_i]_V) - \phi^\sharp([x_i]_V) \in \Gamma(V, \phi_*((d_1^1 \mathcal{N}_1)^2)), \quad (95)$$

and since  $\phi_*((d_1^1 \mathcal{N}_1)^2)$  is a soft sheaf, there are

$$\alpha_i \in \Gamma(Y, \phi_*((d_1^1 \mathcal{N}_1)^2)) \quad (96)$$

extending  $\psi^\sharp([x_i]_V) - \phi^\sharp([x_i]_V)$ . Let  $\gamma_i := \phi^\sharp([x_i]_Y) + \alpha_i$  and consider the sheaves

$$(d_1^1)^{-1}(\nu_j^*(\gamma_1, \dots, \gamma_n) - d_1^1(\phi^\sharp([y_j]_Y))) \subseteq \phi_*(d_2^2 \mathcal{N}_2 + \mathcal{N}_1^2). \quad (97)$$

These are soft sheaves as well, and hence there are

$$\beta_j \in \Gamma(Y, \phi_*(d_2^2 \mathcal{N}_2 + \mathcal{N}_1^2)), \quad (98)$$

extending

$$\psi^\sharp([y_j]) - \phi^\sharp([y_j]) \in \Gamma(V, \phi_*(d_2^2 \mathcal{N}_2 + \mathcal{N}_1^2)), \quad (99)$$

and such that

$$d_1^1(\phi^\sharp([y_j]_Y) + \beta_j) = \nu_j^*(\gamma_1, \dots, \gamma_n). \quad (100)$$

Now we define an element of  $\mathcal{F}(Y)$  as follows:

$$[x_i]_Y \longmapsto \phi^\sharp([x_i]_Y) + \alpha_i, \quad [y_j]_Y \longmapsto \phi^\sharp([y_j]_Y) + \beta_j. \quad (101)$$

Clearly this is an extension of  $\psi^\sharp$ . ■

To use lemma 3 we need to modify Lemma 2, so that the codomains are not necessarily open Kuranishi neighbourhoods, but closed ones. We do this in the following lemma. The proof is straightforward.

**Lemma 4** *Let  $\mathbf{X}, \mathbf{Y} \in \mathbf{G}_{ft}$ , let  $\iota : Z \subseteq Y$  be a closed subset, and let*

$$\mathbf{Y}|_Z := (Z, \iota^{-1}(\mathcal{O}_{\bullet, Y})). \quad (102)$$

*Suppose that the map*

$$\mathbf{T} : \text{Hom}_{\mathbf{G}_{ft}}(\mathbf{X}, \mathbf{Y}) \longrightarrow \text{Hom}_{\underline{\mathbf{G}}_{ft}}(\mathbf{T}(\mathbf{X}), \mathbf{T}(\mathbf{Y})) \quad (103)$$

*is surjective. Then*

$$\mathbf{T} : \text{Hom}_{\mathbf{G}_{ft}}(\mathbf{X}, \mathbf{Y}|_Z) \longrightarrow \text{Hom}_{\underline{\mathbf{G}}_{ft}}(\mathbf{T}(\mathbf{X}), \mathbf{T}(\mathbf{Y}|_Z)) \quad (104)$$

*is surjective as well.*

Now we are ready to show that  $\mathbf{T}$  is almost 1-full. The following proposition makes this statement precise.

**Proposition 12** *Let  $\mathbf{X} \in \underline{\mathbf{G}}_{ft}$ ,  $\mathbf{Y} \in \underline{\mathbf{Man}}_{st}$ , and let*

$$\underline{\phi} : \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{T}(\mathbf{Y}) \quad (105)$$

*be a 1-morphism between the corresponding truncations. There is a morphism  $\Phi : \mathbf{X} \rightarrow \mathbf{Y}$ , s.t.*

$$\mathbf{T}(\Phi) \cong \underline{\phi}. \quad (106)$$



**Proof:** By assumption  $\mathbf{Y}$  is the homotopy equalizer in the diagram

$$\mathcal{M} \xrightleftharpoons[\sigma]{0} E \quad (107)$$

where  $\mathcal{M}$  is a manifold of finite type,  $E$  is a vector bundle over  $\mathcal{M}$ , and  $0, \sigma$  are sections of  $E$ . Therefore, according to Proposition 8, we can assume that

$$\mathbf{Y} = \mathbf{Sp}(\{C^\infty(E^{\times_k \mathcal{M}})\}_{k \geq 0}). \quad (108)$$

Since  $\mathcal{M}$  is second countable, we can cover  $\mathcal{M}$  with a countable family of closed subsets  $\{\mathcal{M}_i\}$ , s.t. over each  $\mathcal{M}_i$  the bundle  $E$  is trivial. Then we have  $\mathbf{Y} = \bigcup_{i \in \mathbb{N}} \mathbf{Y}_i$ , where  $\mathbf{Y}_i$  is the 0-locus of  $\sigma : \mathcal{M}_i \rightarrow E_i$ , and hence each  $\mathbf{Y}_i$  is a closed subscheme of a standard Kuranishi neighbourhood. Let  $\{\mathbf{X}_i\}$  be the pre-images of  $\{\mathbf{Y}_i\}$ .

Now we define a partially ordered set  $P$  as follows. Elements of  $P$  are morphisms

$$\Phi_k : \bigcup_{1 \leq i \leq k} \mathbf{X}_i \longrightarrow \bigcup_{1 \leq i \leq k} \mathbf{Y}_i, \quad k \in \mathbb{N} \cup \{\infty\}, \quad (109)$$

lifting the corresponding restriction of  $\underline{\phi}$ , and  $\Phi_k \geq \Phi_l$  if  $k \geq l$ , and the restriction of  $\Phi_k$  to  $\bigcup_{1 \leq i \leq l} \mathbf{X}_i$  equals  $\Phi_l$ . It is clear that the set  $P$  satisfies the conditions of Zorn lemma, and hence it has maximal elements. If the maximal element has index  $\infty$  (and in particular an  $\infty$ -indexed element exists), we are done, since we have found a lift of  $\underline{\phi}$ .

Suppose a maximal element is  $\Phi_k$ , with  $k < \infty$ . From Lemma 2 and Lemma 4 we know that

$$Hom_{\mathbf{G}_{ft}}(\mathbf{X}_{k+1}, \mathbf{Y}_{k+1}) \longrightarrow Hom_{\underline{\mathbf{G}}_{ft}}(\mathbf{T}(\mathbf{X}_{k+1}), \mathbf{T}(\mathbf{Y}_{k+1})) \quad (110)$$

is surjective, hence  $\underline{\phi}|_{\mathbf{X}_{k+1}}$  has a lift. From Lemma 3 we know that we can choose a lift that agrees with  $\Phi_k$ , restricted to  $(\bigcup_{1 \leq i \leq k} \mathbf{X}_i) \cap \mathbf{X}_{k+1}$ . Thus  $\Phi_k$  cannot be a maximal element. ■

It remains to show that  $\mathbf{T}$  detects equivalences. This is done in the following proposition.

**Proposition 13** *Let  $\Phi : \mathbf{X} \rightarrow \mathbf{Y}$  be a morphism in  $\underline{\mathbf{Man}}_{ft}$ , s.t.  $\mathbf{T}(\Phi)$  is an equivalence. Then  $\Phi$  is a weak equivalence.*

**Proof:** For  $\Phi$  to be a weak equivalence is a local property, since  $\mathbf{T}(\Phi)$  being an equivalence already ensures that the underlying map of topological spaces is a homeomorphism. Therefore, we can assume that  $\mathbf{X}, \mathbf{Y}$  are germs of derived manifolds.

So let  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{m+n}$ ,  $\nu' : \mathbb{R}^{n'} \rightarrow \mathbb{R}^{m'+n'}$  be some sections of trivial bundles, and suppose  $\mathbf{X}, \mathbf{Y}$  are germs of  $\{\nu = 0\}$ ,  $\{\nu' = 0\}$  respectively. We can always find weakly equivalent minimal presentations, i.e. those with minimal  $n$  and  $n'$ . For minimal  $n, n'$  one can show, as in [Jo12] section 5.3, that  $\mathbf{T}(\Phi) : \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{T}(\mathbf{Y})$  being an equivalence implies that  $\Phi$  is an isomorphism. ■

Put altogether, Propositions 11, 12, 13 give the following statement.

**Theorem 1** *The truncation 2-functor*

$$\mathbf{T} : \underline{\mathbf{Man}}_{st} \longrightarrow \underline{Man}_{st} \quad (111)$$

*induces a full and essentially surjective 1-functor between the 1-categories, obtained by identifying isomorphic 1-morphisms. Moreover, the map between the equivalence classes of objects is a bijection.*

This 1-functor is, however, not faithful, as shown in the following example. Let  $\nu : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be defined by

$$\nu^*(u_1) := x^2, \quad \nu^*(u_2) := y^2, \quad \nu^*(u_3) := xy, \quad (112)$$

where  $x, y$  and  $u_1, u_2, u_3$  are the coordinates. Let  $E := \mathbb{R}^5 \rightarrow \mathbb{R}^2$  be the trivial bundle, and let  $\mathbf{Y}$  be the homotopy equalizer of

$$\mathbb{R}^2 \begin{array}{c} \xrightarrow{0} \\ \xRightarrow{\nu} \end{array} E. \quad (113)$$

According to Proposition 8 we can write  $\mathbf{Y}$  as  $\mathbf{Sp}(A_\bullet)$ , where

$$A_\bullet := \{C^\infty(E^{\times_{\mathbb{R}^3}^k})\}_{k \geq 0}. \quad (114)$$

It is easy to see that  $u_1 u_2 - u_3^2 \in \text{Ker}(d_0^1) \cap \text{Ker}(d_1^1) \subset A_1$ . Moreover, the class of  $u_1 u_2 - u_3^2$  in  $\pi_1(A_\bullet)$  is not trivial. Yet,  $u_1 u_2 - u_3^2$  vanishes in the truncation to  $d$ -manifolds, since obviously

$$u_1 u_2 - u_3^2 \in (\text{Ker}(d_0^1))^2 \subset A_1. \quad (115)$$

Let  $\mathbf{X}$  be the homotopy equalizer of

$$\mathbb{R}^0 \begin{array}{c} \xrightarrow{0} \\ \xRightarrow{0} \end{array} \mathbb{R}, \quad (116)$$

and let  $t$  be the coordinate on  $\mathbb{R}$ . We define two morphisms  $\Phi, \Psi : \mathbf{Y} \rightarrow \mathbf{X}$  as follows:

$$\Phi : t \longmapsto 0, \quad \Psi : t \longmapsto u_1 u_2 - u_3^2. \quad (117)$$

Since the class of  $u_1 u_2 - u_3^2$  in  $\pi_1(A_\bullet)$  is not trivial, clearly  $\Phi$  and  $\Psi$  are not homotopic in  $\mathbf{Man}_{ft}$ , yet  $\mathbf{T}(\Phi) = \mathbf{T}(\Psi)$ .

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